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# Multi-Variable White Noise Functions: Standard Setup Revisited

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## 1 Introduction

In the recent development of white noise theory the framework proposed by Cochran–Kuo–Sengupta [4] has become more important for their characterization theorems, see also [1]. In fact, much attention has been paid to characterization theorems for the test functions  $\mathcal{W}$ , for the generalized functions  $\mathcal{W}^*$ , for white noise operators  $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$  and for  $\mathcal{L}(\mathcal{W}, \mathcal{W})$ . As was pointed out first by Chung–Chung–Ji [2], those characterization theorems are related each other, however, the statements are not unified because their objects are different so far as we are concerned with a single CKS-space over a particular underlying Gelfand triple. In this paper, using the *standard setup of white noise calculus* proposed by Hida–Obata–Saitô [6] and by Obata [12], we unify those characterization theorems into a single statement.

Let  $\mathcal{S}_A(U)$  and  $\mathcal{S}_B(V)$  be countable Hilbert nuclear spaces constructed from  $L^2(U)$  and  $L^2(V)$  in the standard manner, respectively, see §2. Then, for two weight sequences  $\alpha = \{\alpha(n)\}_{n=0}^\infty$  and  $\beta = \{\beta(n)\}_{n=0}^\infty$  we consider CKS-spaces of test white noise functions defined by

$$\mathcal{U} \equiv \Gamma_\alpha(\mathcal{S}_A(U)), \quad \mathcal{V} \equiv \Gamma_\beta(\mathcal{S}_B(V)).$$

We assume condition (H1) for both  $\mathcal{S}_A(U)$  and  $\mathcal{S}_B(V)$  and conditions (A1)–(A4) for  $\alpha$  and  $\beta$ . These conditions are described in §§2–3. Then the main result is stated in the following

**Theorem 1.1** *For a continuous operator  $\Xi \in \mathcal{L}(\mathcal{U}, \mathcal{V})$  put*

$$\Theta(\xi, \eta) = \langle \Xi \phi_\xi, \phi_\eta \rangle, \quad \xi \in \mathcal{S}_A(U), \quad \eta \in \mathcal{S}_B(V), \quad (1)$$

*where  $\phi_\xi$  and  $\phi_\eta$  are exponential vectors in  $\mathcal{U}$  and  $\mathcal{V}$ , respectively. Then,  $\Theta$  satisfies the following two conditions:*

- (O1)  $\Theta$  is a Gâteaux-entire function on  $\mathcal{S}_A(U) \times \mathcal{S}_B(V)$ ;
- (O2) for any  $p \geq 0$  there exist  $q \geq 0$  and  $C \geq 0$  such that

$$|\Theta(\xi, \eta)|^2 \leq C G_\alpha(|\xi|_{p+q}^2) G_{1/\beta}(|\eta|_{-p}^2), \quad \xi \in \mathcal{S}_A(U), \quad \eta \in \mathcal{S}_B(V),$$

*where  $G_\alpha$  and  $G_{1/\beta}$  are exponential generating functions of the weight sequences  $\alpha$  and  $1/\beta$ , respectively.*

Conversely, if a  $\mathbf{C}$ -valued function  $\Theta$  defined on  $\mathcal{S}_A(U) \times \mathcal{S}_B(V)$  fulfills conditions (O1) and (O2), there exists a unique continuous operator  $\Xi \in \mathcal{L}(\mathcal{U}, \mathcal{V})$  satisfying (1).

Specializing the underlying spaces  $\mathcal{S}_A(U)$  and  $\mathcal{S}_B(V)$ , we obtain the characterization theorems mentioned at the beginning as corollaries of our main theorem. Moreover, our theorem yields characterization theorems for multi-variable white noise functions in a highly general form. It is noteworthy that multi-variable white noise functions have become more important in applications [13]. Further development is expected together with another type of characterization theorems based on Bargmann–Segal spaces, which has been also extensively studied along with complex white noise [7].

## 2 Standard Construction of an Underlying Gelfand Triple

We assemble some notions and results in [12]. Let  $U$  be a topological space equipped with a  $\sigma$ -finite Borel measure  $\nu$  and consider the complex Hilbert space  $L^2(U) = L^2(U, \nu)$ . Let  $A$  be a selfadjoint operator in  $L^2(U)$  such that  $\inf \text{Spec}(A) > 0$ . With each  $p \in \mathbf{R}$  we associate a Hilbert space  $E_p(U)$  with a norm defined by

$$|\xi|_p = |A^p \xi|_{L^2(U)}, \quad p \in \mathbf{R}.$$

More precisely, for  $p \geq 0$ ,  $E_p(U)$  consists of  $\xi \in L^2(U)$  satisfying  $|\xi|_p < \infty$  and  $E_{-p}(U)$  is the completion of  $L^2(U)$  with respect to the norm  $|\cdot|_{-p}$ . Thus we come to a countable Hilbert space:

$$\mathcal{S}_A(U) = \text{proj} \lim_{p \rightarrow \infty} E_p(U).$$

The strong dual space of  $\mathcal{S}_A(U)$  is identical to the inductive limit:

$$\mathcal{S}_A^*(U) = \text{ind} \lim_{p \rightarrow \infty} E_{-p}(U)$$

and we come to a rigging:

$$\mathcal{S}_A(U) \subset L^2(U) \subset \mathcal{S}_A^*(U). \quad (2)$$

The canonical  $\mathbf{C}$ -bilinear form on  $\mathcal{S}_A^*(U) \times \mathcal{S}_A(U)$  is denoted by  $\langle \cdot, \cdot \rangle$ . Then by definition the norm of  $L^2(U)$  is given by  $|\xi|_0^2 = \langle \bar{\xi}, \xi \rangle$ .

**Lemma 2.1** *A countable Hilbert space  $\mathcal{S}_A(U)$  defined as above is nuclear if and only if there exists  $r > 0$  such that  $A^{-r}$  is of Hilbert–Schmidt type.*

As for a spectral property of  $A$  we consider

(H1)  $\inf \text{Spec}(A) > 1$  and  $A^{-r}$  is of Hilbert–Schmidt type for some  $r > 0$ .

The first condition is taken into account beforehand. It follows from (H1) that

$$\|A^{-1}\|_{\text{OP}} < 1, \quad \lim_{r \rightarrow \infty} \|A^{-r}\|_{\text{HS}} = 0. \quad (3)$$

**Definition** A countable Hilbert nuclear space  $\mathcal{S}_A(U)$  constructed from  $L^2(U)$  by means of a selfadjoint operator  $A$  satisfying (H1) is called *standard*.

As is suggested by (2), it is natural to consider  $\mathcal{S}_A(U)$  and  $\mathcal{S}_A^*(U)$  are spaces of test functions and generalized functions (or distributions) on  $U$ , respectively. However, it is not clear at all whether the delta function (evaluation map) is a member of  $\mathcal{S}_A^*(U)$ . On the contrary, continuity of a test function does not follow automatically. By construction each element of  $\mathcal{S}_A(U)$  is merely a function on  $U$  which is determined up to  $\nu$ -null functions. We thus come to:

(H2) for each function  $\xi \in \mathcal{S}_A(U)$  there exists a unique continuous function  $\tilde{\xi}$  on  $U$  such that  $\xi(u) = \tilde{\xi}(u)$  for  $\nu$ -a.e.  $u \in U$ .

Once this condition is satisfied, we consider  $\mathcal{S}_A(U)$  always as a space of continuous functions on  $U$  and we do not use the exclusive symbol  $\tilde{\xi}$ . Under (H2) we put two more hypotheses to keep a delta function in  $\mathcal{S}_A^*(U)$ :

(H3) for each  $u \in U$  the evaluation map  $\delta_u : \xi \mapsto \xi(u)$ ,  $\xi \in \mathcal{S}_A(U)$ , is a continuous linear functional, i.e.,  $\delta_u \in \mathcal{S}_A^*(U)$ ;

(H4) the map  $u \mapsto \delta_u \in \mathcal{S}_A^*(U)$ ,  $u \in U$ , is continuous with respect to the strong dual topology of  $\mathcal{S}_A^*(U)$ .

Conditions similar to (H1)–(H4) were also discussed by Kubo–Takenaka [11]. These hypotheses are essential for topological arguments of  $\mathcal{S}_A(U)$ . Here we recall the following

**Proposition 2.2** *Let  $\mathcal{S}_A(U)$  be a standard countable Hilbert space and let  $\xi_n \in \mathcal{S}_A(U)$ ,  $n = 1, 2, \dots$ , be a sequence converging to 0 in  $\mathcal{S}_A(U)$ . If (H2) and (H3) are satisfied, then the sequence converges pointwisely, i.e.,  $\lim_{n \rightarrow \infty} \xi_n(u) = \xi(u)$  for any  $u \in U$ . Moreover, if (H4) is satisfied in addition, the pointwise convergence is uniform on every compact subset of  $U$ .*

**Proposition 2.3** *Let  $\mathcal{S}_A(U)$  be a standard countable Hilbert space satisfying conditions (H2) and (H3). Then,*

$$\|A^{-r}\|_{\text{HS}}^2 = \int_U |\delta_u|_{-r}^2 \nu(du) = \sum_{j=0}^{\infty} \lambda_j^{-2r} < \infty,$$

where  $1 < \lambda_1 \leq \lambda_2 \leq \dots$  are the eigenvalues of  $A$ .

Recall that for two selfadjoint operators  $A_i$  in a Hilbert space  $H_i$  ( $i = 1, 2$ ) their tensor product  $A_1 \otimes A_2$  becomes a selfadjoint operator in  $H_1 \otimes H_2$  in a canonical way. Moreover, if  $\inf \text{Spec}(A_i) > 0$  for  $i = 1, 2$ , then  $\inf \text{Spec}(A_1 \otimes A_2) > 0$  as well.

**Proposition 2.4** *Let  $\mathcal{S}_A(U)$  and  $\mathcal{S}_B(V)$  be standard countable Hilbert nuclear spaces. Then the canonical isomorphism  $L^2(U \times V) \cong L^2(U) \otimes L^2(V)$  induces a topological isomorphism:*

$$\mathcal{S}_{A \otimes B}(U \times V) \cong \mathcal{S}_A(U) \otimes \mathcal{S}_B(V). \quad (4)$$

Moreover, if both  $\mathcal{S}_A(U)$  and  $\mathcal{S}_B(V)$  satisfy hypotheses (H2)–(H4), so does  $\mathcal{S}_{A \otimes B}(U \times V)$ .

Useful sufficient conditions for (H2)–(H4) are known, see [12, §1.4]. In fact, these conditions are essential to formulate quantum white noise  $\{a_u, a_u^*; u \in U\}$ , however, in this paper we do not go into this subject.

### 3 Conditions for Weight Sequences

After Asai-Kubo-Kuo [1] we introduce some general notion for positive sequences. A sequence  $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$  of positive numbers is called *log-concave* if

$$\alpha(n)\alpha(n+2) \leq \alpha(n+1)^2, \quad n = 0, 1, 2, \dots$$

Two positive sequences  $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$  and  $\beta = \{\beta(n)\}_{n=0}^{\infty}$  are called *equivalent* if there exist positive constants  $K_1, K_2, M_1, M_2 > 0$  such that

$$K_1 M_1^n \alpha(n) \leq \beta(n) \leq K_2 M_2^n \alpha(n), \quad n = 0, 1, 2, \dots$$

For a positive sequence  $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$  we consider the following conditions:

(A1)  $\alpha(0) = 1$  and  $\inf_{n \geq 0} \sigma^n \alpha(n) > 0$  for some  $\sigma \geq 1$ ;

(A2)  $\lim_{n \rightarrow \infty} \left\{ \frac{\alpha(n)}{n!} \right\}^{1/n} = 0$ ;

(A3)  $\alpha$  is equivalent to a positive sequence  $\gamma = \{\gamma(n)\}$  such that  $\left\{ \frac{\gamma(n)}{n!} \right\}$  is log-concave;

(A4)  $\alpha$  is equivalent to another positive sequence  $\gamma = \{\gamma(n)\}$  such that  $\left\{ \frac{1}{n! \gamma(n)} \right\}$  is log-concave.

For example,  $(n!)^\beta$  with  $0 \leq \beta < 1$  and the Bell numbers of order  $k$  satisfy the above conditions [4].

The exponential generating functions of  $\alpha$  and  $1/\alpha$  are defined by

$$G_\alpha(t) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n, \quad G_{1/\alpha}(t) = \sum_{n=0}^{\infty} \frac{1}{n! \alpha(n)} t^n,$$

respectively. Both are entire holomorphic functions by (A1) and (A2). We next define

$$\begin{aligned} \tilde{G}_\alpha(t) &\equiv \sum_{n=0}^{\infty} t^n \frac{n^{2n}}{n! \alpha(n)} \left\{ \inf_{\tau > 0} \frac{G_\alpha(\tau)}{\tau^n} \right\}, \\ \tilde{G}_{1/\alpha}(t) &\equiv \sum_{n=0}^{\infty} t^n \frac{n^{2n} \alpha(n)}{n!} \left\{ \inf_{\tau > 0} \frac{G_{1/\alpha}(\tau)}{\tau^n} \right\}. \end{aligned}$$

It is known [1] that (A3) and (A4) are necessary and sufficient conditions respectively for  $\tilde{G}_\alpha$  and for  $\tilde{G}_{1/\alpha}$  to have positive radii of convergence. These functions will play a crucial role in norm estimates. Moreover, the next fact is known [1].

**Lemma 3.1** Assume that  $\alpha = \{\alpha(n)\}$  satisfies (A1)–(A4).

(1) There exists a constant  $C_{\alpha 1} > 0$  such that

$$\alpha(n)\alpha(m) \leq C_{\alpha 1}^{n+m} \alpha(n+m), \quad n, m = 0, 1, 2, \dots$$

(2) There exists a constant  $C_{\alpha 2} > 0$  such that

$$\alpha(n+m) \leq C_{\alpha 2}^{n+m} \alpha(n)\alpha(m), \quad n, m = 0, 1, 2, \dots$$

(3) There exists a constant  $C_{\alpha 3} > 0$  such that

$$\alpha(n) \leq C_{\alpha 3}^m \alpha(m), \quad 0 \leq n \leq m.$$

Then, by a simple calculation we have

**Proposition 3.2** *Let  $\alpha = \{\alpha(n)\}$  be as above and  $G_\alpha(t)$  the exponential generating function. Then, for  $s, t \geq 0$  we have:*

- (1)  $G_\alpha(0) = 1$  and  $G_\alpha(s) \leq G_\alpha(t)$  for  $s \leq t$ .
- (2)  $G_\alpha(s)G_\alpha(t) \leq G_\alpha(C_{\alpha 1}(s+t))$ .
- (3)  $G_\alpha(s+t) \leq G_\alpha(C_{\alpha 2}s)G_\alpha(C_{\alpha 2}t)$ .
- (4)  $e^s G_\alpha(t) \leq G_\alpha(C_{\alpha 3}(s+t))$ , in particular,  $e^t \leq G_\alpha(C_{\alpha 3}t)$ .

## 4 Standard CKS-Space

Suppose we are given a standard, countable Hilbert nuclear space:

$$\mathcal{S}_A(U) = \text{proj} \lim_{p \rightarrow \infty} E_p(U)$$

and a positive sequence  $\alpha = \{\alpha(n)\}$  satisfying (A1)–(A4). We first form a weighted Fock space:

$$\Gamma_\alpha(E_p(U)) = \left\{ \phi = (f_n)_{n=0}^\infty; f_n \in E_p(U)^{\widehat{\otimes} n}, \|\phi\|_p^2 \equiv \sum_{n=0}^\infty n! \alpha(n) |f_n|_p^2 < \infty \right\},$$

where  $E_p(U)^{\widehat{\otimes} n}$  stands for the  $n$ -fold symmetric tensor power, and then take its projective limit:

$$\mathcal{U} = \Gamma_\alpha(\mathcal{S}_A(U)) = \text{proj} \lim_{p \rightarrow \infty} \Gamma_\alpha(E_p(U)).$$

Since  $\mathcal{U}$  becomes a countable Hilbert nuclear space, we obtain a Gelfand triple:

$$\mathcal{U} = \Gamma_\alpha(\mathcal{S}_A(U)) \subset \Gamma(L^2(U)) \subset \Gamma_\alpha(\mathcal{S}_A(U))^* = \mathcal{U}^*, \quad (5)$$

where the middle space is the usual Boson Fock space, i.e., a weighted Fock space with weight sequence  $\alpha(n) \equiv 1$ . We may consider (5) as a variant of “second quantization” of an underlying Gelfand triple:

$$\mathcal{S}_A(U) \subset L^2(U) \subset \mathcal{S}_A^*(U).$$

We call (5) a *standard CKS-space* after [4].

The topology of  $\mathcal{U}$  is defined by the family of norms:

$$\|\phi\|_{p,+}^2 = \sum_{n=0}^\infty n! \alpha(n) |f_n|_p^2, \quad \phi = (f_n) \in \mathcal{U}, \quad p \geq 0.$$

The canonical  $\mathbf{C}$ -bilinear form on  $\mathcal{U}^* \times \mathcal{U}$  is denoted by  $\langle\langle \cdot, \cdot \rangle\rangle$ . Then

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \quad \Phi = (F_n) \in \mathcal{U}^*, \quad \phi = (f_n) \in \mathcal{U},$$

and it holds that

$$|\langle\langle \Phi, \phi \rangle\rangle| \leq \|\Phi\|_{-p,-} \|\phi\|_{p,+},$$

where

$$\|\Phi\|_{-p,-}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} |F_n|_{-p}^2, \quad \Phi = (F_n) \in \mathcal{U}^*.$$

We also note that

$$\Gamma_{\alpha}(\mathcal{S}_A(U))^* \cong \operatorname{ind} \lim_{p \rightarrow \infty} \Gamma_{1/\alpha}(E_{-p}(U)),$$

where  $\Gamma_{\alpha}(\mathcal{S}_A(U))^*$  carries the strong dual topology and  $\cong$  stands for a topological linear isomorphism.

Conditions (A1)–(A4) are sorted out by Asai–Kubo–Kuo [1] from many similar ones that have been introduced to keep “nice” properties of a CKS-space. On the other hand, it is also possible to start with a generating function  $G_{\alpha}$  or another function controlling growth rate. This reversed approach is concise and useful for some questions [1], [5], [8], however, we prefer to the explicit description for our later calculation.

By the Wiener–Itô–Segal isomorphism the Boson Fock space  $\Gamma(L^2(U))$  is realized as an  $L^2$ -space over a Gaussian space. In that sense  $\Gamma_{\alpha}(\mathcal{S}_A(U))$  is a space of functions on the Gaussian space. By a parallel argument with [12, §3.2] one can show easily that  $\Gamma_{\alpha}(\mathcal{S}_A(U))$  satisfies conditions (H2)–(H4). Thus, for example, white noise delta functions are defined as white noise distributions.

## 5 Proof of Main Theorem

First we recall some notation used in the statement of Theorem 1.1. For each  $\xi \in \mathcal{S}_A(U)$  we put

$$\phi_{\xi} = \left( 1, \frac{\xi}{1!}, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots \right).$$

Then  $\phi_{\xi} \in \mathcal{U}$  and is called an *exponential vector* or a *coherent vector*. Let  $\mathfrak{X}_1, \dots, \mathfrak{X}_n$  be complex topological vector spaces in general. Then a  $\mathbf{C}$ -valued function  $F$  defined on  $\mathfrak{X}_1 \times \dots \times \mathfrak{X}_n$  is called *Gâteaux-entire* if the function

$$z \mapsto F(\xi_1, \dots, \xi_k + z\xi', \dots, \xi_n)$$

is entire holomorphic in  $z \in \mathbf{C}$  for any choice of  $\xi_1 \in \mathfrak{X}_1, \dots, \xi_n \in \mathfrak{X}_n$  and  $\xi' \in \mathfrak{X}_k$ ,  $1 \leq k \leq n$ .

Now we start with the following

**Lemma 5.1** Let  $F : \mathcal{S}_A(U) \rightarrow \mathbb{C}$  be a Gâteaux-entire function. Assume that there exist an entire function  $G$  on  $\mathbb{C}$  and  $p \in \mathbb{R}$  such that

$$|F(\xi)|^2 \leq G(|\xi|_p^2), \quad \xi \in \mathcal{S}_A(U).$$

Then for any  $n \geq 0$  the Gâteaux derivative

$$F_n(\xi_1, \dots, \xi_n) = \frac{1}{n!} D_{\xi_1} \dots D_{\xi_n} F(0) \quad (6)$$

becomes a continuous  $n$ -linear form on  $\mathcal{S}_A(U)$  satisfying

$$\|F_n\|_{-(p+s)}^2 \leq \left(\frac{n^n}{n!}\right)^2 \left\{ \inf_{\tau > 0} \frac{G(\tau)}{\tau^n} \right\} \|A^{-s}\|_{\text{HS}}^{2n}. \quad (7)$$

**PROOF.** 1°. It is a standard result that  $F_n$  is a (not necessarily continuous)  $n$ -linear form on  $\mathcal{S}_A(U)$ .

2°. The Taylor expansion of an entire function  $z \mapsto F(z\xi)$  is given by

$$F(z\xi) = \sum_{n=0}^{\infty} F_n(\xi, \dots, \xi) z^n, \quad \xi \in \mathcal{S}_A(U).$$

3°. By Cauchy's integral formula we obtain

$$|F_n(\xi, \dots, \xi)| \leq |\xi|_p^n \left\{ \inf_{\tau > 0} \frac{G(\tau)}{\tau^n} \right\}^{1/2}.$$

4°. By the polarization formula we have

$$\sup\{|F_n(\xi_1, \dots, \xi_n)|; |\xi_1|_p \leq 1, \dots, |\xi_n|_p \leq 1\} \leq \frac{n^n}{n!} \left\{ \inf_{\tau > 0} \frac{G(\tau)}{\tau^n} \right\}^{1/2}.$$

5°. Let  $1 < \lambda_0 \leq \lambda_1 \leq \dots$  be the eigenvalues of  $A$  and  $\{e_j\}_{j=0}^{\infty}$  the corresponding eigenvectors which form a complete orthonormal basis of  $L^2(U)$ . Then

$$\begin{aligned} \|F_n\|_{-(p+s)}^2 &= \sum_{j_1, \dots, j_n=0}^{\infty} |F_n(e_{j_1}, \dots, e_{j_n})|^2 \lambda_{j_1}^{-2(p+s)} \dots \lambda_{j_n}^{-2(p+s)} \\ &= \sum_{j_1, \dots, j_n=0}^{\infty} |F_n(\lambda_{j_1}^{-p} e_{j_1}, \dots, \lambda_{j_n}^{-p} e_{j_n})|^2 \lambda_{j_1}^{-2s} \dots \lambda_{j_n}^{-2s} \\ &\leq \left(\frac{n^n}{n!}\right)^2 \left\{ \inf_{\tau > 0} \frac{G(\tau)}{\tau^n} \right\} \|A^{-s}\|_{\text{HS}}^{2n}. \end{aligned}$$

This completes the proof. ■

**Lemma 5.2** Let  $F : \mathcal{S}_A(U) \rightarrow \mathbb{C}$  be a Gâteaux-entire function. Assume that there exist constants  $C \geq 0$  and  $p \in \mathbb{R}$  such that

$$|F(\xi)|^2 \leq CG_\alpha(|\xi|_p^2), \quad \xi \in \mathcal{S}_A(U).$$

For each  $n \geq 0$  denote by  $F_n$  the  $n$ -th Gâteaux derivative defined by (6). Then  $\Phi = (F_n) \in \Gamma_\alpha(\mathcal{S}_A(U))^*$  and we have

$$\|\Phi\|_{-(p+s),-}^2 \leq C\tilde{G}_\alpha(\|A^{-s}\|_{\text{HS}}^2).$$



PROOF. By the definition of norms and (7) we have

$$\begin{aligned} \|\Phi\|_{-(p+s),-}^2 &= \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} \|F_n\|_{-(p+s)}^2 \\ &\leq \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} \left(\frac{n^n}{n!}\right)^2 \left\{ \inf_{\tau>0} \frac{CG_\alpha(\tau)}{\tau^n} \right\} \|A^{-s}\|_{\text{HS}}^{2n} \\ &= C\tilde{G}_\alpha(\|A^{-s}\|_{\text{HS}}^2), \end{aligned}$$

as desired. ■

Recall that  $\tilde{G}_\alpha$  has a positive radius of convergence. Then  $\tilde{G}_\alpha(\|A^{-s}\|_{\text{HS}}^2) < \infty$  for all sufficiently large  $s > 0$  because  $\lim_{s \rightarrow \infty} \|A^{-s}\|_{\text{HS}} = 0$  by (3). In a similar manner we have

**Lemma 5.3** *Let  $F : \mathcal{S}_A(U) \rightarrow \mathbf{C}$  be a Gâteaux-entire function. Assume that there exist constants  $C \geq 0$  and  $p \in \mathbf{R}$  such that*

$$|F(\xi)|^2 \leq CG_{1/\alpha}(|\xi|_p^2), \quad \xi \in \mathcal{S}_A(U).$$

*For each  $n \geq 0$  let  $F_n$  be the  $n$ -th Gâteaux derivative defined by (6). Then  $\Phi = (F_n) \in \Gamma_\alpha(\mathcal{S}_A(U))^*$  and we have*

$$\|\Phi\|_{-(p+s),+}^2 \leq C\tilde{G}_{1/\alpha}(\|A^{-s}\|_{\text{HS}}^2).$$

**Remark** Lemma 5.2 is the so-called characterization theorem for white noise distributions first shown by Potthoff–Streit [15] for a particular CKS-space called the Hida–Kubo–Takenaka space. The essence of their proof, however, remains invariable though a few variants have been discussed during the recent development of white noise theory. Lemma 5.3 implies immediately characterization theorem for white noise test functions  $\Gamma_\alpha(\mathcal{S}_A(U))$ . In this connection see also Theorem 6.1.

PROOF OF THEOREM 1.1. Fix  $\eta \in \mathcal{S}_B(V)$  and we consider

$$F_\eta(\xi) = \Theta(\xi, \eta), \quad \xi \in \mathcal{S}_A(U).$$

Then by Lemma 5.2 there exists  $\Phi_\eta \in \mathcal{U}^*$  such that

$$F_\eta(\xi) = \langle\langle \Phi_\eta, \phi_\xi \rangle\rangle, \quad \text{i.e.,} \quad \Theta(\xi, \eta) = \langle\langle \Phi_\eta, \phi_\xi \rangle\rangle,$$

and

$$\|\Phi_\eta\|_{-(p+q+s),-}^2 \leq C\tilde{G}_\alpha(\|A^{-s}\|_{\text{HS}}^2)G_{1/\beta}(|\eta|_{-p}^2).$$

Next, for a fixed  $\phi \in \mathcal{U}$  we consider

$$H_\phi(\eta) = \langle\langle \Phi_\eta, \phi \rangle\rangle, \quad \eta \in \mathcal{S}_B(V).$$

Obviously,

$$\begin{aligned} |H_\phi(\eta)|^2 &\leq \|\Phi_\eta\|_{-(p+q+s),-}^2 \|\phi\|_{p+q+s,+}^2 \\ &\leq C\tilde{G}_\alpha(\|A^{-s}\|_{\text{HS}}^2)G_{1/\beta}(|\eta|_{-p}^2) \|\phi\|_{p+q+s,+}^2. \end{aligned}$$

Moreover,  $H_\phi(\eta + z\eta')$  is a compact uniform limit of a sequence of entire functions which are linear combinations of

$$H_{\phi_\xi}(\eta + z\eta') = \langle\langle \Phi_{\eta+z\eta'}, \phi_\xi \rangle\rangle = \Theta(\xi, \eta + z\eta'),$$

where  $\xi$  runs over  $\mathcal{S}_A(U)$ , and hence  $H_\phi$  is Gâteaux-entire. Applying Lemma 5.3 to  $H_\phi$  we find  $\Psi_\phi \in \mathcal{V}^*$  such that

$$H_\phi(\eta) = \langle\langle \Psi_\phi, \phi_\eta \rangle\rangle, \quad \eta \in \mathcal{S}_B(V)$$

and

$$\|\Psi_\phi\|_{-(-p+t),+}^2 \leq C\tilde{G}_\alpha(\|A^{-s}\|_{\text{HS}}^2)\tilde{G}_{1/\beta}(\|A^{-t}\|_{\text{HS}}^2)\|\phi\|_{p+q+s,+}^2. \quad (8)$$

Define a linear map  $\Xi : \mathcal{U} \rightarrow \mathcal{V}$  by  $\Xi\phi = \Psi_\phi$ . Then (8) implies that

$$\|\Xi\phi\|_{p-t,+}^2 \leq C\tilde{G}_\alpha(\|A^{-s}\|_{\text{HS}}^2)\tilde{G}_{1/\beta}(\|A^{-t}\|_{\text{HS}}^2)\|\phi\|_{p+q+s,+}^2,$$

which proves that  $\Xi$  is continuous. Since

$$\langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle = H_{\phi_\xi}(\eta) = \langle\langle \Phi_\eta, \phi_\xi \rangle\rangle = \Theta(\xi, \eta),$$

this  $\Xi \in \mathcal{L}(\mathcal{U}, \mathcal{V})$  is what we searched for. ■

In fact, as the above proof is a simple combination of [12, §3.6] and [2], nothing new is required essentially. A few variants of the proof are easily obtained following the arguments [1], [3], [4], [5], [9], [10].

## 6 Unification of Traditional Characterization Theorems

Since the exponential vectors  $\{\phi_\xi; \xi \in E_{\mathbf{C}}\}$  are linearly independent and span a dense subspace of  $\mathcal{U}$ , they play a fundamental role in specifying white noise functions and white noise operators. In practice, the most important are the  $S$ -transform of  $\Phi \in \mathcal{U}^*$  defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle, \quad \xi \in \mathcal{S}_A(U),$$

and the *symbol* of  $\Xi \in \mathcal{L}(\mathcal{U}, \mathcal{U}^*)$  defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \quad \xi, \eta \in \mathcal{S}_A(U).$$

The traditional characterization theorems are mentioned as follows.

**Theorem 6.1** *A  $\mathbf{C}$ -valued function  $F$  defined on  $\mathcal{S}_A(U)$  is the  $S$ -transform of some  $\Phi \in \mathcal{U}^*$  if and only if*

(F1)  *$F$  is Gâteaux-entire;*

(F2) *there exist  $C \geq 0$  and  $p \geq 0$  such that*

$$|F(\xi)|^2 \leq CG_\alpha(|\xi|_p^2), \quad \xi \in E_{\mathbf{C}}.$$

Moreover,  $\Phi \in \mathcal{U}$  if and only if (F1) and

(F3) for any  $p \geq 0$  there exist  $C \geq 0$  such that

$$|F(\xi)|^2 \leq CG_{1/\alpha}(|\xi|_{-p}^2), \quad \xi \in E_{\mathbf{C}}.$$

**Theorem 6.2** A  $\mathbf{C}$ -valued function  $\Theta$  defined on  $\mathcal{S}_A(U) \times \mathcal{S}_A(U)$  is the symbol of an operator  $\Xi \in \mathcal{L}(\mathcal{U}, \mathcal{U}^*)$  if and only if

(O1)  $\Theta$  is Gâteaux-entire;

(O2) there exist constant numbers  $C \geq 0$  and  $p \geq 0$  such that

$$|\Theta(\xi, \eta)|^2 \leq CG_{\alpha}(|\xi|_p^2)G_{\alpha}(|\eta|_p^2), \quad \xi, \eta \in \mathcal{S}_A(U).$$

Moreover,  $\Xi \in \mathcal{L}(\mathcal{U}, \mathcal{U})$  if and only if (O1) and

(O3) for any  $p \geq 0$  there exist constant numbers  $C \geq 0$  and  $q \geq 0$  such that

$$|\Theta(\xi, \eta)|^2 \leq CG_{\alpha}(|\xi|_{p+q}^2)G_{1/\alpha}(|\eta|_{-p}^2), \quad \xi, \eta \in \mathcal{S}_A(U).$$

We may regard the zero space  $\{0\}$  also as a standard countable Hilbert space though there is no underlying topological space  $U$  or rather we may understand that  $\mathcal{S}(\emptyset) = \{0\}$ . Then  $\Gamma_{\alpha}(\mathcal{S}(\emptyset))$  is one-dimensional space consisting only scalar multiples of the vacuum vector  $\phi_0$ . As for Theorem 6.1, if we set  $\mathcal{S}_B(V) = \{0\}$  in Theorem 1.1, the statement is characterization of  $\mathcal{S}_A(U)^*$ . If we set  $\mathcal{S}_A(U) = \{0\}$ , the statement is characterization of  $\mathcal{S}_B(V)$ . The second part of Theorem 6.2 is immediate from Theorem 1.1 by setting  $\mathcal{S}_A(U) = \mathcal{S}_B(V)$ . We need some discussion to derive the first part of Theorem 6.2.

**Lemma 6.3** If  $\mathcal{S}_A(U)$  and  $\mathcal{S}_B(V)$  are standard countable Hilbert nuclear spaces, so is  $\mathcal{S}_{A \oplus B}(U \cup V)$  and  $\mathcal{S}_{A \oplus B}(U \cup V) \cong \mathcal{S}_A(U) \oplus \mathcal{S}_B(V)$ .

**Lemma 6.4** If  $\mathcal{S}_A(U)$  is a standard countable Hilbert nuclear space, so is  $\mathcal{S}_{A \otimes I}(U \times T)$  for any finite discrete space  $T$  equipped with counting measure. Moreover,  $\mathcal{S}_{A \otimes I}(U \times T) \cong \mathcal{S}_A(U) \otimes \mathbf{C}^n$ , where  $|T| = n$ .

The above results are straightforward. In general, for two Hilbert spaces  $H_1$  and  $H_2$  there is a canonical unitary isomorphism

$$\mathcal{T} : \Gamma(H_1) \otimes \Gamma(H_2) \rightarrow \Gamma(H_1 \oplus H_2)$$

specified by the correspondence of exponential vectors  $\mathcal{T}(\phi_{\xi} \otimes \phi_{\eta}) = \phi_{\xi \oplus \eta}$ . In fact, for

$$\phi_{\xi_1, \dots, \xi_j} = (0, \dots, 0, \xi_1 \hat{\otimes} \dots \hat{\otimes} \xi_j, 0, \dots), \quad \psi_{\eta_1, \dots, \eta_k} = (0, \dots, 0, \eta_1 \hat{\otimes} \dots \hat{\otimes} \eta_k, 0, \dots),$$

$\mathcal{T}(\phi_{\xi_1, \dots, \xi_j} \otimes \psi_{\eta_1, \dots, \eta_k})$  is given as

$$\mathcal{T}(\phi_{\xi_1, \dots, \xi_j} \otimes \psi_{\eta_1, \dots, \eta_k}) = (0, \dots, 0, h_{j+k}, 0, \dots),$$

where

$$h_{j+k} = (\xi_1 \oplus 0) \hat{\otimes} \dots \hat{\otimes} (\xi_j \oplus 0) \hat{\otimes} (0 \oplus \eta_1) \hat{\otimes} \dots \hat{\otimes} (0 \oplus \eta_k).$$

For arbitrary  $\phi = (0, \dots, 0, f_j, 0, \dots) \in \Gamma(H_1)$  and  $\psi = (0, \dots, 0, g_k, 0, \dots) \in \Gamma(H_2)$ ,  $\mathcal{T}(\phi \otimes \psi)$  is given by a bilinear map  $h_{j,k} : H_1^{\otimes j} \times H_2^{\otimes k} \rightarrow (H_1 \oplus H_2)^{\otimes(j+k)}$  in such a way that

$$\mathcal{T}(\phi \otimes \psi) = (0, \dots, 0, h_{j,k}(f_j, g_k), 0, \dots). \quad (9)$$

For this  $h_{j,k}$  we have by Fourier expansion

$$|h_{j,k}(f_j, g_k)|_{(H_1 \oplus H_2)^{\otimes(j+k)}}^2 = \frac{j!k!}{(j+k)!} |f_j|_{H_1^{\otimes j}}^2 |g_k|_{H_2^{\otimes k}}^2. \quad (10)$$

**Lemma 6.5** *The canonical isomorphism  $\Gamma(L^2(U) \oplus L^2(V)) \cong \Gamma(L^2(U)) \otimes \Gamma(L^2(V))$  induces a topological isomorphism:*

$$\Gamma_\alpha(\mathcal{S}_A(U) \oplus \mathcal{S}_B(V)) \cong \Gamma_\alpha(\mathcal{S}_A(U)) \otimes \Gamma_\alpha(\mathcal{S}_B(V)).$$

**PROOF.** Let  $\mathcal{T}$  be the canonical isomorphism from  $\Gamma(L^2(U)) \otimes \Gamma(L^2(V))$  onto  $\Gamma(L^2(U) \oplus L^2(V))$  described above. For  $\phi = (f_j) \in \Gamma(L^2(U))$  and  $\psi = (g_k) \in \Gamma(L^2(V))$ , we set  $\mathcal{T}(\phi \otimes \psi) = (h_n) \in \Gamma(L^2(U) \oplus L^2(V))$ . Then by (9) we have

$$h_n = \sum_{j+k=n} h_{j,k}(f_j, g_k).$$

Since the right hand side is an orthogonal sum, using (10) we come to

$$|h_n|_p^2 = \sum_{j+k=n} |h_{j,k}(f_j, g_k)|_p^2 = \sum_{j+k=n} \frac{j!k!}{(j+k)!} |f_j|_p^2 |g_k|_p^2.$$

Hence

$$\|\mathcal{T}(\phi \otimes \psi)\|_{p,+}^2 = \sum_{n=0}^{\infty} n! \alpha(n) |h_n|_p^2 = \sum_{j,k=0}^{\infty} j!k! \alpha(j+k) |f_j|_p^2 |g_k|_p^2. \quad (11)$$

Then by Lemma 3.1 (2),

$$\|\mathcal{T}(\phi \otimes \psi)\|_{p,+}^2 \leq \sum_{j=0}^{\infty} j! \alpha(j) C_{\alpha 2}^j |f_j|_p^2 \sum_{k=0}^{\infty} k! \alpha(k) C_{\alpha 2}^k |g_k|_p^2. \quad (12)$$

Choose  $q \geq 0$  such that  $C_{\alpha 2} \|A^{-1}\|_{\text{OP}}^{2q} \leq 1$  and  $C_{\alpha 2} \|B^{-1}\|_{\text{OP}}^{2q} \leq 1$ . Then (12) becomes

$$\|\mathcal{T}(\phi \otimes \psi)\|_{p,+}^2 \leq \sum_{j=0}^{\infty} j! \alpha(j) |f_j|_{p+q}^2 \sum_{k=0}^{\infty} k! \alpha(k) |g_k|_{p+q}^2 = \|\phi\|_{p+q,+}^2 \|\psi\|_{p+q,+}^2,$$

that is,

$$\|\mathcal{T}(\phi \otimes \psi)\|_{p,+} \leq \|\phi\|_{p+q,+} \|\psi\|_{p+q,+}. \quad (13)$$

In a similar manner, applying Lemma 3.1 (1) to (11), we obtain

$$\|\mathcal{T}(\phi \otimes \psi)\|_{p,+} \geq \|\phi\|_{p-r,+} \|\psi\|_{p-r,+}, \quad (14)$$

where  $r \geq 0$  is taken in such a way that  $C_{\alpha 1} \|A^{-1}\|_{\text{OP}}^{2r} \leq 1$  and  $C_{\alpha 1} \|B^{-1}\|_{\text{OP}}^{2r} \leq 1$ . The assertion then follows from (13) and (14).  $\blacksquare$

**Remark** For Hilbert spaces  $H_1$  and  $H_2$  there is no isomorphism between  $\Gamma_\alpha(H_1 \oplus H_2)$  and  $\Gamma_\alpha(H_1) \otimes \Gamma_\alpha(H_2)$  for a general  $\alpha$ .

**Lemma 6.6** *Let  $\mathcal{S}_A(U)$  be a standard countable Hilbert nuclear space and let  $T$  be a discrete space of  $n$  points with counting measure. Then the isomorphism*

$$\Gamma(L^2(U \times T)) \cong \Gamma(L^2(U)) \otimes \cdots \otimes \Gamma(L^2(U)) \quad (n \text{ times})$$

*induces a topological isomorphism*

$$\Gamma_\alpha(\mathcal{S}_{A \otimes I}(U \times T)) \cong \Gamma_\alpha(\mathcal{S}_A(U)) \otimes \cdots \otimes \Gamma_\alpha(\mathcal{S}_A(U)) \quad (n \text{ times}).$$

PROOF. Note first that

$$L^2(T) = \mathcal{S}_I(T) \cong \mathbb{C}^n.$$

It then follows from Proposition 2.4 that the canonical isomorphism

$$L^2(U \times T) \cong L^2(U) \otimes L^2(T) \cong L^2(U) \oplus \cdots \oplus L^2(U) \quad (n \text{ times}) \quad (15)$$

induces a topological isomorphism:

$$\mathcal{S}_{A \otimes I}(U \times T) \cong \mathcal{S}_A(U) \otimes \mathcal{S}_I(T) \cong \mathcal{S}_A(U) \oplus \cdots \oplus \mathcal{S}_A(U) \quad (n \text{ times}).$$

On the other hand, from (15) we see that

$$\Gamma(L^2(U \times T)) \cong \Gamma(L^2(U)) \otimes \cdots \otimes \Gamma(L^2(U)) \quad (n \text{ times})$$

and from Lemma 6.5

$$\Gamma_\alpha(\mathcal{S}_{A \otimes I}(U \times T)) \cong \Gamma_\alpha(\mathcal{S}_A(U)) \otimes \cdots \otimes \Gamma_\alpha(\mathcal{S}_A(U)) \quad (n \text{ times}).$$

The assertion is then clear. ■

We go back to Theorem 1.1. By Lemma 6.6 the statement of Theorem 1.1 remains valid if  $U$  is replaced with  $U \times \{1, 2\}$  and  $V$  with  $\emptyset$ . Moreover,

$$\begin{aligned} \mathcal{L}(\Gamma_\alpha(\mathcal{S}_{A \otimes I}(U \times \{1, 2\})), \mathbb{C}) &\cong \mathcal{L}(\Gamma_\alpha(\mathcal{S}_A(U)) \otimes \Gamma_\alpha(\mathcal{S}_A(U)), \mathbb{C}) \\ &\cong \{\Gamma_\alpha(\mathcal{S}_A(U)) \otimes \Gamma_\alpha(\mathcal{S}_A(U))\}^* \\ &\cong \mathcal{L}(\Gamma_\alpha(\mathcal{S}_A(U)), \Gamma_\alpha(\mathcal{S}_A(U))^*). \end{aligned}$$

Thus, in this case, Theorem 1.1 is reduced to characterizaion of white noise operators  $\mathcal{L}(\Gamma_\alpha(\mathcal{S}_A(U)), \Gamma_\alpha(\mathcal{S}_A(U))^*)$ .

In order to complete the reduction we need to discuss conditions (O1) and (O2), see Theorem 6.2. Let  $T = \{1, 2, \dots, n\}$ . By the isomorphism  $\mathcal{S}_{A \otimes I}(U \times T) \cong \mathcal{S}_A(U) \otimes \mathbb{C}^n$  described in Lemma 6.4 we come to

$$\mathcal{S}_{A \otimes I}(U \times T) \cong \mathcal{S}_A(U) \oplus \cdots \oplus \mathcal{S}_A(U) \quad (n \text{ times}).$$

For  $(\xi_1, \dots, \xi_n) \in \mathcal{S}_A(U) \oplus \cdots \oplus \mathcal{S}_A(U)$  the corresponding element  $\xi \in \mathcal{S}_{A \otimes I}(U \times T)$  is given by

$$\xi(u, j) = \xi_j(u)$$

Then, there is a one-to-one correspondence between functions on  $\mathcal{S}_{A \otimes I}(U \times T)$  and on  $\mathcal{S}_A(U) \times \cdots \times \mathcal{S}_A(U)$  ( $n$  times) given by

$$\Theta(\xi) = F(\xi_1, \dots, \xi_n).$$

**Lemma 6.7** *Notations being as above,  $\Theta$  is Gâteaux-entire on  $\mathcal{S}_{A \otimes I}(U \times T)$  if and only if so is  $F$  on  $\mathcal{S}_A(U) \times \cdots \times \mathcal{S}_A(U)$  ( $n$  times).*

PROOF. We need only to recall Hartogs' theorem of holomorphy. ■

**Lemma 6.8** *Notations being as above, if there exist  $C \geq 0$  and  $p \in \mathbf{R}$  such that*

$$|\Theta(\xi)|^2 \leq C G_\alpha(|\xi|_p^2), \quad \xi \in \mathcal{S}_{A \otimes I}(U \times T),$$

*then there exists  $q \geq 0$  such that*

$$|F(\xi_1, \dots, \xi_n)|^2 \leq C \prod_{j=1}^n G_\alpha(|\xi_j|_{p+q}^2), \quad \xi_1, \dots, \xi_n \in \mathcal{S}_A(U).$$

*Conversely, if there exist  $C \geq 0$  and  $p \in \mathbf{R}$  such that*

$$|F(\xi_1, \dots, \xi_n)|^2 \leq C \prod_{j=1}^n G_\alpha(|\xi_j|_p^2), \quad \xi_1, \dots, \xi_n \in \mathcal{S}_A(U),$$

*then there exists  $q \geq 0$  such that*

$$|\Theta(\xi)|^2 \leq C G_\alpha(|\xi|_{p+q}^2), \quad \xi \in \mathcal{S}_{A \otimes I}(U \times T).$$

PROOF. This is a simple consequence of Proposition 3.2 (2) and (3). ■

With the help of Lemmas 6.7 and 6.8 we see immediately that conditions (O1) and (O2) in Theorem 1.1 in the case where  $U$  and  $V$  are replaced with  $U \times \{1, 2\}$  and  $\emptyset$ , respectively, coincide with the usual ones in Theorem 6.2.

## 7 Characterization Theorems for Multi-Variable Case

Let us start with a single CKS-space:

$$\mathcal{U} = \Gamma_\alpha(\mathcal{S}_A(U)) \subset \Gamma(L^2(U)) \subset \Gamma_\alpha(\mathcal{S}_A(U))^* = \mathcal{U}^*. \quad (16)$$

We are interested in multi-variable functions defined on  $\mathcal{S}_A(U) \times \cdots \times \mathcal{S}_A(U)$  ( $n$ -times), in particular, of the forms:

$$F(\xi_1, \dots, \xi_m) = \langle\langle \Phi, \phi_{\xi_1} \otimes \cdots \otimes \phi_{\xi_m} \rangle\rangle, \quad (17)$$

$$G(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n) = \langle\langle \Xi(\phi_{\xi_1} \otimes \cdots \otimes \phi_{\xi_m}), \phi_{\eta_1} \otimes \cdots \otimes \phi_{\eta_n} \rangle\rangle, \quad (18)$$

where  $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n \in \mathcal{S}_A(U)$ , and  $\Phi \in (\mathcal{U}^{\otimes m})^*$ ,  $\Xi \in \mathcal{L}(\mathcal{U}^{\otimes m}, (\mathcal{U}^{\otimes n})^*)$ . It is clear that the functions defined in (17) and (18) are Gâteaux-entire.

The following results are immediate corollaries of Theorem 1.1 with the help of Lemmas 6.7 and 6.8.

**Theorem 7.1** *A Gâteaux-entire function  $F : \mathcal{S}_A(U)^m \rightarrow \mathbf{C}$  is expressed in the form (17) with  $\Phi \in (\mathcal{U}^{\otimes m})^*$  if and only if there exist constant numbers  $C \geq 0$  and  $p \geq 0$  such that*

$$|F(\xi_1, \dots, \xi_m)|^2 \leq C \prod_{j=1}^m G_\alpha(|\xi_j|_p^2), \quad \xi_1, \dots, \xi_m \in \mathcal{S}_A(U).$$

**Theorem 7.2** A Gâteaux-entire function  $F : \mathcal{S}_A(U)^m \rightarrow \mathbf{C}$  is expressed in the form (17) with  $\Phi \in \mathcal{U}^{\otimes m}$  if and only if for any  $p \geq 0$  there exists  $C \geq 0$  such that

$$|F(\xi_1, \dots, \xi_m)|^2 \leq C \prod_{j=1}^m G_{1/\alpha}(|\xi_j|_{-p}^2), \quad \xi_1, \dots, \xi_m \in \mathcal{S}_A(U).$$

**Theorem 7.3** A Gâteaux-entire function  $G : \mathcal{S}_A(U)^{m+n} \rightarrow \mathbf{C}$  is expressed in the form (18) with  $\Xi \in \mathcal{L}(\mathcal{U}^{\otimes m}, (\mathcal{U}^{\otimes n})^*)$  if and only if there exist  $C \geq 0$  and  $p \geq 0$  such that

$$|G(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n)|^2 \leq C \prod_{j=1}^m G_\alpha(|\xi_j|_p^2) \prod_{k=1}^n G_\alpha(|\eta_k|_p^2),$$

for  $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n \in \mathcal{S}_A(U)$ .

In that case, since  $\mathcal{L}(\mathcal{U}^{\otimes m}, (\mathcal{U}^{\otimes n})^*) \cong (\mathcal{U}^{\otimes(m+n)})^*$ , we may choose an operator  $\Xi$  from  $\mathcal{L}(\mathcal{U}^{\otimes m'}, (\mathcal{U}^{\otimes n'})^*)$  whenever  $m' + n' = m + n$ .

**Theorem 7.4** A Gâteaux-entire function  $G : \mathcal{S}_A(U)^{m+n} \rightarrow \mathbf{C}$  is expressed in the form (18) with  $\Xi \in \mathcal{L}(\mathcal{U}^{\otimes m}, \mathcal{U}^{\otimes n})$  if and only if for any  $p \geq 0$  there exist  $C \geq 0$  and  $q \geq 0$  such that

$$|G(\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n)|^2 \leq C \prod_{j=1}^m G_\alpha(|\xi_j|_{p+q}^2) \prod_{k=1}^n G_{1/\alpha}(|\eta_k|_{-p}^2),$$

for  $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n \in \mathcal{S}_A(U)$ .

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